

ANALYTIC CLONES

MARTIN GOLDSTERN

ABSTRACT. We use a method from descriptive set theory to investigate the two complete clones above the unary clone on a countable set.

1. INTRODUCTION. KNOWN RESULTS

An “operation” on a set X is a function $f : X^n \rightarrow X$, for some $n \in \mathbb{N} \setminus \{0\}$. If f is such an “ n -ary operation”, g_1, \dots, g_n are k -ary, then the “composition” $f(g_1, \dots, g_n)$ is defined naturally:

$$f(g_1, \dots, g_n)(\vec{x}) = f(g_1(\vec{x}), \dots, g_n(\vec{x})) \quad \text{for all } \vec{x} \in X^k$$

A clone on a set X is a set \mathcal{C} of operations which contains all the projections and is closed under composition. (Alternatively, \mathcal{C} is a clone on X if \mathcal{C} is the set of term functions of some universal algebra over X .)

The family of all clones forms a complete algebraic lattice $Cl(X)$ with greatest element $\mathcal{O} = \bigcup_{n=1}^{\infty} \mathcal{O}^{(n)}$, where $\mathcal{O}^{(n)} = X^{X^n}$ is the set of all n -ary operations on X . (In this paper, the underlying set X will always be the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers.)

The coatoms of this lattice $Cl(X)$ are called “precomplete clones” or “maximal clones” on X .

For any set $\mathcal{C} \subseteq \mathcal{O}$ we write $\langle \mathcal{C} \rangle$ for the smallest clone containing \mathcal{C} . In particular, $\langle \mathcal{O}^{(1)} \rangle$ is the set of all functions $\pi \circ f$, where $f : X \rightarrow X$ is arbitrary and $\pi : X^n \rightarrow X$ is a projection to one coordinate. However, to lighten the notation we will identify $\mathcal{O}^{(1)}$ (the set of all unary functions) with $\langle \mathcal{O}^{(1)} \rangle$ (the set of all “essentially” unary functions).

For singleton sets X the lattice $Cl(X)$ is trivial; for $|X| = 2$ the lattice $Cl(X)$ is countable, and well understood (“Post’s lattice”). For $|X| \geq 3$, $Cl(X)$ is uncountable. For infinite X , $Cl(X)$ has $2^{|X|}$ elements, and there are even $2^{2^{|X|}}$ precomplete clones on X .

In this paper we are interested in the interval $[\mathcal{O}^{(1)}, \mathcal{O}]$ of the clone lattice on a countable set X . It will turn out that methods from descriptive set theory are useful to describe the complexity of several interesting clones in this interval, and also the overall structure of the interval.

For simplicity we concentrate on binary clones, i.e., clones generated by binary functions. Equivalently, we can define a binary clone to be a set \mathcal{C} of functions $f : \mathbb{N}^2 \rightarrow \mathbb{N}$ which contains the two projections and is closed under composition: if $f, g, h \in \mathcal{C}$, then also the function $f(g, h)$ (mapping (x, y) to $f(g(x, y), h(x, y))$) is in \mathcal{C} .

The set of binary clones, $Cl^{(2)}(X)$, also forms a complete algebraic lattice.

Date: 2004-04-11.

2000 *Mathematics Subject Classification.* primary 08A40; secondary 03E15.

Occasionally we will remark on how to modify the definitions or theorems for the case of “full” clones, i.e., for clones that are not necessarily generated by binary functions. (In some cases this generalization is trivial, in other cases it is nontrivial but known, and in some cases it is still open.)

By [1] (see also [2]), we know that there are exactly 2 precomplete binary clones above $\mathcal{O}^{(1)}$, which we call T_1 and T_2 (see below). It is known that the interval $[\mathcal{O}^{(1)}, \mathcal{O}^{(2)}]$ of binary clones is dually atomic, so it can be written

$$[\mathcal{O}^{(1)}, \mathcal{O}^{(2)}] = [\mathcal{O}^{(1)}, \mathsf{T}_1] \cup [\mathcal{O}^{(1)}, \mathsf{T}_2] \cup \{\mathcal{O}^{(2)}\},$$

i.e., every binary clone above $\mathcal{O}^{(1)}$ other than $\mathcal{O}^{(2)}$ itself is contained in T_1 or in T_2 .

So we will have to investigate the intervals $[\mathcal{O}^{(1)}, \mathsf{T}_1]$ and $[\mathcal{O}^{(1)}, \mathsf{T}_2]$. We will see that these two structures are very different, and that this difference can be traced back to a difference in “complexity” of the two binary clones T_1 and T_2 .

More precisely, T_1 is a Borel set, while T_2 is a complete coanalytic set. We will see that T_1 is finitely generated over $\mathcal{O}^{(1)}$, but T_2 is not countably generated over $\mathcal{O}^{(1)}$.

1.1. Definition. A function $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ is called “almost unary”, if at least one of the following holds:

- (x) There is a function $F : \mathbb{N} \rightarrow \mathbb{N}$ such that $\forall x \forall y : f(x, y) \leq F(x)$.
- (y) There is a function $F : \mathbb{N} \rightarrow \mathbb{N}$ such that $\forall x \forall y : f(x, y) \leq F(y)$.

We let T_1 be the set of all binary functions which are almost unary. It is easy to see that T_1 is a binary clone containing $\mathcal{O}^{(1)}$.

1.2. Definition. Let $B \subseteq \mathcal{O}^{(2)}$. The set $\text{Pol}(B)$ is defined as

$$\bigcup_{k=1}^{\infty} \{f \in \mathcal{O}^{(k)} : \forall g_1, \dots, g_k \in B : f(g_1, \dots, g_k) \in B\}$$

(Background and a more general definition of Pol can be found in [6].)

1.3. Fact. $\text{Pol}(B)$ is a clone. If B is a binary clone, then $\text{Pol}(B) \cap \mathcal{O}^{(2)} = B$.

1.4. Definition. Let $\Delta := \{(x, y) \in \mathbb{N} \times \mathbb{N} : x > y\}$, $\nabla := \{(x, y) : x < y\}$.

For $S_1, S_2 \subseteq \mathbb{N}$ we let $\Delta_{S_1, S_2} := \Delta \cap (S_1 \times S_2)$. We define ∇_{S_1, S_2} similarly.

If S_1, S_2 are infinite subsets of \mathbb{N} , and $g : \Delta_{S_1, S_2} \rightarrow \mathbb{N}$ or $g : \nabla_{S_1, S_2} \rightarrow \mathbb{N}$, then we say that g is “canonical” iff one of the following holds:

- (1) g is constant
- (2) There is a 1-1 function $G : S_1 \rightarrow \mathbb{N}$ such that

$$\forall (x, y) \in \text{dom}(g) : g(x, y) = G(x)$$

- (3) There is a 1-1 function $G : S_2 \rightarrow \mathbb{N}$ such that

$$\forall (x, y) \in \text{dom}(g) : g(x, y) = G(y)$$

- (4) g is 1-1.

The “type” of g is one of the labels “constant”, “x”, “y”, or “1-1”, respectively.

Let $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$. We say that f is canonical on $S_1 \times S_2$ iff both functions $f \upharpoonright \Delta_{S_1, S_2}$ and $f \upharpoonright \nabla_{S_1, S_2}$ are canonical (but not necessarily of the same type), and moreover:

Either the ranges of $f \upharpoonright \Delta_{S_1, S_2}$ and $f \upharpoonright \nabla_{S_1, S_2}$ are disjoint,
or $S_1 = S_2$, and $f(x, y) = f(y, x)$ for all $x, y \in S_1$.

The following fact is a consequence of Ramsey’s theorem, see [2]. It was originally proved in a slightly different formulation already in [1].

1.5. *Fact.* Let $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$. Then there are infinite sets S_1, S_2 such that f is canonical on $S_1 \times S_2$.

Moreover, for any infinite sets S_1, S_2 we can find infinite $S'_1 \subseteq S_1, S'_2 \subseteq S_2$ such that f is canonical on $S'_1 \times S'_2$.

1.6. *Definition.* Let $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$. We say that f is “nowhere injective”, if:

whenever f is canonical on $S_1 \times S_2$, then neither $f \upharpoonright \Delta_{S_1, S_2}$ nor $f \upharpoonright \nabla_{S_1, S_2}$ is 1-1.

We let T_2 be the set of all nowhere injective functions. Using fact 1.5, it is easy to check that T_2 is a binary clone; clearly T_2 contains $\mathcal{O}^{(1)}$. (More precisely, T_2 contains $\langle \mathcal{O}^{(1)} \rangle \cap \mathcal{O}^{(2)}$.)

1.7. *Theorem* (Gavrilov [1]). T_1 and T_2 are precomplete binary clones, and every binary clone containing $\mathcal{O}^{(1)}$ is either contained in one of $\mathsf{T}_1, \mathsf{T}_2$, or equal to the clone of all binary functions.

[For the non-binary case: $\text{Pol}(\mathsf{T}_1)$ and $\text{Pol}(\mathsf{T}_2)$ are precomplete clones, and every clone $\supseteq \mathcal{O}^{(1)}$ is either $= \mathcal{O}$, or $\subseteq \text{Pol}(\mathsf{T}_1)$, or $\subseteq \text{Pol}(\mathsf{T}_2)$.]

We will prove the following:

- (Section 3) T_1 is finitely generated over $\mathcal{O}^{(1)}$, so the interval $[\mathcal{O}^{(1)}, \mathsf{T}_1]$ in the lattice of binary clones is dually atomic.
In fact, the interval contains a unique coatom: $\mathsf{T}_1 \cap \mathsf{T}_2$.
- (Section 4) T_2 is neither finitely nor countably generated over $\mathcal{O}^{(1)}$.
 $\mathsf{T}_1 \cap \mathsf{T}_2$ is a coatom in the interval $[\mathcal{O}^{(1)}, \mathsf{T}_2]$. (Easy)
Any clone which is a Borel set (or even an analytic set) cannot be a coatom in this interval.

1.8. *Acknowledgement.* I am grateful to J. Jezek, K. Kearnes, R. Pöschel, A. Romanowska, A. Szendrei and R. Willard for inviting me to Bela Csakany’s birthday conference (Szeged, 2002), at which I first presented the main ideas from this paper.

2. DESCRIPTIVE SET THEORY

We collect a few facts and notions from descriptive set theory. (For motivation, history, details and proofs see the textbooks by Moschovakis [4] or Kechris [3].)

Let X be a countable set (usually $X = \mathbb{N}$, or $X = \mathbb{N}^k$), and Y a finite or countable set, $|Y| \geq 2$ (usually $Y = \mathbb{N}$, or $Y = 2 := \{0, 1\}$). Y^X is the space of all functions from X to Y .

We equip Y with the discrete topology, Y^X and Y^{X^n} with the product topology, and $\bigcup_{n=1}^{\infty} Y^{X^n}$ with the sum topology. All these spaces are “Polish spaces”, i.e., they are separable and carry a (natural) complete metric.

The family of Borel sets is the smallest family \mathcal{B} that contains all open sets and is closed under complements and countable unions (equivalently: contains all open sets and all closed sets, and is closed under countable unions and countable intersections).

A function f between two topological spaces is called a Borel function iff the preimage of any Borel set under f is again a Borel set.

A *finite sequence on Y* is a tuple $(a_0, \dots, a_{n-1}) \in Y^n$. If $s \in Y^k$ and $t \in Y^n$ are finite sequences, $k < n$, then we write $s \triangleleft t$ iff s is an initial segment of t .

We write $Y^{<\omega} := \bigcup_{n \in \mathbb{N}} Y^n$ for the set of all finite sequences on Y . If Y is countable, then also $Y^{<\omega}$ is countable.

We can identify $\mathcal{P}(Y^{<\omega})$, the power set of $Y^{<\omega}$, with the set $2^{Y^{<\omega}}$ of all characteristic functions, so also $\mathcal{P}(Y^{<\omega})$ carries a natural topology.

A “tree on Y ” is a set $T \subseteq \bigcup_{n \in \mathbb{N}} Y^n$ of finite sequences which is downward closed, i.e.,

$$\text{whenever } t \in T, s \triangleleft t, \text{ then also } s \in T$$

The set of all trees is easily seen to be a closed subset of $\mathcal{P}(Y^{<\omega})$.

For any tree T on Y we call $f \in Y^{\mathbb{N}}$ a *branch* of T iff $\forall n : f \upharpoonright n \in T$. (Here we write $f \upharpoonright n$ for $(f(0), \dots, f(n-1))$.)

We write $[T]$ for the set of all branches of T .

It is easy to see that $[T]$ is always a closed set in $Y^{\mathbb{N}}$, and that every closed set $\subseteq Y^{\mathbb{N}}$ is of the form $[T]$ for some tree T .

We call a tree T *well-founded* if $[T] = \emptyset$, i.e., if there is no sequence $s_0 \triangleleft s_1 \triangleleft \dots$ of elements of T .

We write **WF** for the set of all well-founded trees.

The class of *analytic sets* is a proper extension of the class of Borel sets. There are several possible equivalent definitions of “analytic”, for example one could choose the equivalence (1) \Leftrightarrow (3) in fact 2.1 as the definition of “analytic”.

2.1. Fact. Let \mathcal{X} be a Polish (=complete metric separable) topological space, $A \subseteq \mathcal{X}$, $C := \mathcal{X} \setminus A$. Then the following are equivalent:

- (1) A is analytic
- (2) C is coanalytic
- (3) $A = \emptyset$, or there is a continuous function $f : \mathbb{N}^{\mathbb{N}} \rightarrow \mathcal{X}$ with $A = f[\mathbb{N}^{\mathbb{N}}]$
- (4) There is a Borel set $B \subseteq \mathbb{N}^{\mathbb{N}}$ and a continuous function $f : \mathbb{N}^{\mathbb{N}} \rightarrow \mathcal{X}$ with $A = f[B]$
- (5) There is a continuous function $f : \mathcal{X} \rightarrow \mathcal{P}(\mathbb{N}^{<\omega})$ such that $C = f^{-1}[\mathbf{WF}]$.
- (6) (Assuming $\mathcal{X} = Y^{\mathbb{N}}$.) There is a set $R \subseteq Y^{<\omega} \times \mathbb{N}^{<\omega}$ such that

$$A = \{f \in Y^{\mathbb{N}} : \exists g \in \mathbb{N}^{\mathbb{N}} \forall n (f \upharpoonright n, g \upharpoonright n) \in R\}$$

The coanalytic sets are just the sets whose complement is analytic. Borel sets are of course both analytic and coanalytic, and the “Separation theorem” states that the converse is true:

Let $A \subseteq \mathcal{X}$ be both analytic and coanalytic. Then A is a Borel set.

Analytic sets have the following closure properties:

- 2.2. *Fact.* (1) All Borel sets are analytic (and coanalytic).
 (2) The countable union or intersection of analytic sets is again analytic. Similarly, the countable union or intersection of coanalytic sets is again analytic.
 (3) The continuous preimage of an analytic set is analytic. The continuous preimage of a coanalytic set is coanalytic.
 (4) The continuous image of an analytic set is analytic. (Note that the continuous image of a Borel set is in general not Borel.)
 (5) In particular, if $C \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is a Borel set, then the set $\{f \in \mathbb{N}^{\mathbb{N}} : \exists g \in \mathbb{N}^{\mathbb{N}} (f, g) \in C\}$ is analytic, and the set $\{f \in \mathbb{N}^{\mathbb{N}} : \forall g \in \mathbb{N}^{\mathbb{N}} (f, g) \in C\}$ is coanalytic.

However, while the Borel sets are closed under complements, the analytic sets are not. There are coanalytic sets which are not analytic, for example the set **WF**.

We call a set $D \subseteq Y^X$ “complete coanalytic” iff

- (1) D is coanalytic
- (2) For any coanalytic set $C \subseteq Y^X$ there is a continuous function $F : Y^X \rightarrow Y^X$ with $C = F^{-1}[D]$.

It is known that the set **WF** is complete coanalytic. In fact, **WF** is the “typical” coanalytic set:

Let D be coanalytic. Then D is complete coanalytic iff there is a continuous function $F : \mathcal{P}(\mathbb{N}^{<\omega}) \rightarrow Y^X$ with $\mathbf{WF} = F^{-1}[D]$.

Equivalently, D is complete coanalytic iff there is a function as above which is defined only on the set of trees.

The existence of coanalytic sets which are not analytic easily implies that a complete coanalytic set can never be analytic.

The following theorem should be read as “analytic sets can never reach ω_1 .”

2.3. *Fact* (Boundedness theorem).

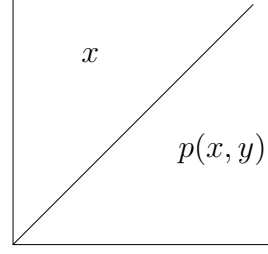
- (1) Every coanalytic set is the union of an increasing ω_1 -chain of Borel sets.
- (2) Let $\mathbf{WF} = \bigcup_{\alpha \in \omega_1} \mathbf{WF}_\alpha$ be an increasing union of Borel sets, and let $A \subseteq \mathbf{WF}$ be Borel (or even analytic).

Then there is $\alpha \in \omega_1$ such that $A \subseteq \mathbf{WF}_\alpha$.

3. CLONES BELOW T_1

3.1. *Definition.* We fix a 1-1 function p from $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N} \setminus \{0\}$. Let χ_Δ and χ_∇ be the characteristic functions of Δ and ∇ , and let $p_\Delta := p \cdot \chi_\Delta$, i.e., $p_\Delta(x, y) = p(x, y)$ for $x > y$, and $= 0$ otherwise.

Similarly, let $p_\nabla := p \cdot \chi_\nabla$.

Properties of q (simplified)

The following is clear:

3.2. *Fact.*

- χ_∇ and χ_Δ are canonical, and in $\mathsf{T}_1 \cap \mathsf{T}_2$.
- p_Δ and p_∇ are in $\mathsf{T}_1 \setminus \mathsf{T}_2$, and are canonical.
- $p \notin \mathsf{T}_1 \cup \mathsf{T}_2$. In fact, the only clone containing $\mathcal{O}^{(1)} \cup \{p\}$ is \mathcal{O} itself.

3.3. *Theorem.* T_1 is generated by $\{p_\Delta\} \cup \mathcal{O}^{(1)}$.

Proof. Let \mathcal{C} be the binary clone generated by $\{p_\Delta\} \cup \mathcal{O}^{(1)}$. We will first find a function $q \in \mathcal{C}$ satisfying

- (1) q is 1-1 on Δ
- (2) $q(x, y) = Q(x)$ on ∇ , for some 1-1 function Q
- (3) $q[\Delta] \cap q[\nabla] = \emptyset$.

Note that any two functions q, q' satisfying these properties will be equivalent, in the sense that there is a unary function u with $q(x, y) = u(q'(x, y))$ for all $(x, y) \in \Delta \cup \nabla$.

Let $P(x) = \max\{p(x, y) : y \leq x\} + 1$, and let

$$q(x, y) := p_\Delta(P(x), p_\Delta(x, y)).$$

Note that this actually means $q(x, y) = p(P(x), p_\Delta(x, y))$, as $P(x) > p_\Delta(x, y)$ for all x, y . So,

$$q(x, y) = \begin{cases} p(P(x), p(x, y)) & \text{for } x > y \\ p(P(x), 0) & \text{for } x \leq y, \end{cases}$$

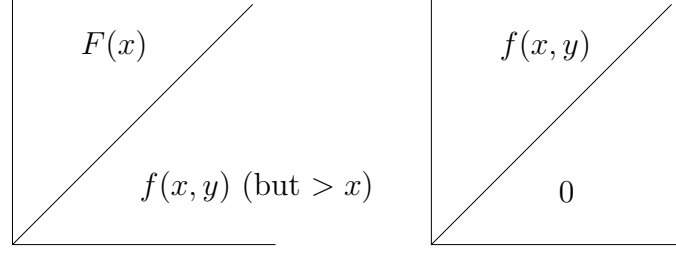
So q satisfies (1)–(3), and $q \in \mathcal{C}$.

We now consider an arbitrary almost unary function f , say $f(x, y) < F(x)$ for all x, y . Wlog we assume $f(x, y) > 0$ for all (x, y) .

Let $p' : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be a 1-1 function satisfying $p'(x, y) > x$ for all x, y .

Define

$$\begin{aligned} f_1(x, y) &= \begin{cases} p'(F(x), f(x, y)) & \text{for } x > y \\ F(x) & \text{for } x \leq y \end{cases} \\ f_2(x, y) &= \begin{cases} 0 & \text{for } x > y \\ f(x, y) & \text{for } x \leq y \end{cases} \end{aligned}$$

Definitions of f_1 and f_2 (simplified)

Then $f_1(x, y) = u_1(q(x, y))$ for some unary u_1 , and $f_2(x, y) = u_2(p_\Delta(y + 1, x))$ for some unary u_2 . So $f_1, f_2 \in \mathcal{C}$.

Let $f'(x, y) := p_\Delta(f_1(x, y), f_2(x, y))$. Now $f_2(x, y) < F(x) \leq f_1(x, y)$ for all x, y , so $f'(x, y) = p(f_1(x, y), f_2(x, y))$.

As $f(x, y)$ can be recovered from the pair $(f_1(x, y), f_2(x, y))$, and hence also from $f'(x, y)$, we conclude that $f(x, y) = v(f'(x, y))$ for some unary v . Hence $f \in \mathcal{C}$. \square

3.4. *Theorem.* If $\mathcal{C} \subseteq \mathbf{T}_1$ is a binary clone containing $\mathcal{O}^{(1)}$, then either $\mathcal{C} = \mathbf{T}_1$, or $\mathcal{C} \subseteq \mathbf{T}_2$.

Hence: $\mathbf{T}_1 \cap \mathbf{T}_2$ is the unique coatom in the interval $[\mathcal{O}^{(1)}, \mathbf{T}_1]$ of binary clones, and every binary clone in this interval (except for \mathbf{T}_1 itself) is included in $\mathbf{T}_1 \cap \mathbf{T}_2$.

Proof. Assume $\mathcal{O}^{(1)} \subseteq \mathcal{C} \subseteq \mathbf{T}_1$, but $\mathcal{C} \not\subseteq \mathbf{T}_2$. So let $f \in \mathcal{C} \setminus \mathbf{T}_2$. So there are 1-1 unary functions u and v such that $f(u(x), v(y))$ is canonical and 1-1 on Δ (or on ∇). So wlog f is canonical and 1-1 on Δ .

Moreover, either f is symmetric, or $\text{ran}(f \upharpoonright \nabla) \cap \text{ran}(f \upharpoonright \Delta) = \emptyset$.

In the first case, the function $f'(x, y) := f(2x, 2y + 1)$ is 1-1 on all of $\mathbb{N} \times \mathbb{N}$, so $\langle \{f'\} \cup \mathcal{O}^{(1)} \rangle = \mathcal{C}$, which contradicts our assumption $\mathcal{C} \subseteq \mathbf{T}_1$.

In the second case, we can find a unary function u such that

$$\forall x, y : u(f(2x, 2y + 1)) = p_\Delta(x, y),$$

so $p_\Delta \in \mathcal{C}$, i.e., $\mathcal{C} = \mathbf{T}_1$. \square

Pinsker [5] has analyzed the interval $(\mathbf{T}_1, \text{Pol}(\mathbf{T}_1))$ of (full) clones, and shown the following:

3.5. *Theorem* (Pinsker). Let $\min_n^+(x_1, \dots, x_n) := x_{\pi(2)}$, where π is any permutation such that $x_{\pi(1)} \leq x_{\pi(2)} \leq \dots \leq x_{\pi(n)}$.

(So $\min_2^+(x, y) = \max(x, y)$, and $\min_3^+(x, y, z)$ is the median of x, y, z .)

Then the clones $\mathcal{M}_n := \langle \mathbf{T}_1 \cup \{\min_n^+\} \rangle$ are all distinct,

$$\mathbf{T}_1 \subsetneq \dots \subsetneq \mathcal{M}_5 \subsetneq \mathcal{M}_4 \subsetneq \mathcal{M}_3 = \text{Pol}(\mathbf{T}_2) \subsetneq \mathcal{M}_2 = \mathcal{C},$$

and every clone in the interval $[\mathbf{T}_1, \text{Pol}(\mathbf{T}_1)]$ is equal to some \mathcal{M}_n .

3.6. *Remark.* So \mathcal{M}_4 is a coatom in the interval $[\mathcal{O}^{(1)}, \text{Pol}(\mathbf{T}_1)]$ in the lattice of all clones. It is also easy to see that $\text{Pol}(\mathbf{T}_1) \cap \text{Pol}(\mathbf{T}_2) = \text{Pol}(\mathbf{T}_1 \cap \mathbf{T}_2)$ is another coatom.

3.7. *Fact.* T_1 is a Borel set.

Proof. The set $\mathsf{T}_1^x := \{f \in \mathcal{O}^{(2)} : \exists F \forall x, y : f(x, y) \leq F(x)\}$ is apparently only Σ_1^1 , but we can rewrite it as

$$\begin{aligned} \mathsf{T}_1^x &= \{f \in \mathcal{O}^{(2)} : \forall x \exists z \forall y : f(x, y) \leq z\} \\ &= \bigcap_{x \in \mathbb{N}} \bigcup_{z \in \mathbb{N}} \bigcap_{y \in \mathbb{N}} \bigcup_{t \leq z} \{f \in \mathcal{O}^{(2)} : f(x, y) = t\}, \end{aligned}$$

which is $F_{\sigma\delta}$.

T_1^y can be defined similarly, and $\mathsf{T}_1 = \mathsf{T}_1^x \cup \mathsf{T}_1^y$. □

3.8. *Remark.* Clearly, $\text{Pol}(\mathsf{T}_1)$ is coanalytic. (See 2.2(5).) By Pinsker's theorem, $\text{Pol}(\mathsf{T}_1) = \langle \mathsf{T}_1 \cup \{\min_3^+\} \rangle$ is finitely generated over T_1 , hence analytic and therefore even Borel. An explicit Borel description can be found in [5].

4. CLONES BELOW T_2

In the previous section we have seen:

4.1. *Theorem.* $\mathsf{T}_1 = \langle \mathcal{O}^{(1)} \cup \{p_\Delta\} \rangle$. Thus, T_1 is finitely generated over $\mathcal{O}^{(1)}$.

The next theorem and its corollaries show that T_2 is not finitely generated over $\mathcal{O}^{(1)}$.

4.2. *Fact.* Let $B \subseteq \mathcal{O}$ be a Borel or analytic set. Then $\langle B \rangle$ is analytic.

Similarly, if $B \subseteq \mathcal{O}^{(2)}$ is a Borel or analytic set, then $\langle B \rangle_{\mathcal{O}^{(2)}}$ (the binary clone generated by B) is analytic.

4.3. *Question.* Is there a Borel set B (perhaps even a closed set? a countable set? a set of the form $\mathcal{O}^{(1)} \cup \{f_1, \dots, f_n\}$?) such that $\langle B \rangle$ is not Borel?

4.4. *Theorem.* T_2 , $\text{Pol}(\mathsf{T}_2)$, $\mathsf{T}_1 \cap \mathsf{T}_2$ and $\text{Pol}(\mathsf{T}_1 \cap \mathsf{T}_2)$ are complete coanalytic sets.

Proof. We will define a continuous map F from the set of all trees $T \subseteq \mathbb{N}^{<\mathbb{N}}$ into T_1 such that

for all T : T is wellfounded iff $F(T) \in \mathsf{T}_2$.

Let $\{s_n : n \in \mathbb{N}\}$ enumerate all finite sequences of natural numbers, with $s_k \triangleleft s_n \Rightarrow k < n$.

For any tree $T \subseteq \{s_n : n \in \mathbb{N}\}$ let $F(T)$ be defined as follows:

$$F(T)(k, n) = \begin{cases} p(k, n) & \text{if } k < n \text{ and } s_k, s_n \in T, s_k \triangleleft s_n \\ 0 & \text{otherwise} \end{cases}$$

Now if $A := \{s_{n_1}, s_{n_2}, \dots\}$ is an infinite branch in T , then $F(T) \upharpoonright \nabla_{A,A}$ is 1-1.

Conversely: Assume $A = \{n_1 < n_2 < \dots\}$, $B = \{m_1 < m_2 < \dots\}$, and $F(T) \upharpoonright \nabla_{A,B}$ is 1-1.

We claim that $s_{n_1} \triangleleft s_{n_2}$. Indeed, for any large enough k we have $F(T)(n_1, m_k) \neq 0$, so $s_{n_1} \triangleleft s_{m_k}$, and similarly $s_{n_2} \triangleleft s_{m_k}$. So $s_{n_1} \triangleleft s_{n_2}$.

Similarly we get $s_{n_1} \triangleleft s_{n_2} \triangleleft s_{n_3} \triangleleft \dots$. □

4.5. *Corollary.* $\text{Pol}(\mathsf{T}_2)$, T_2 , $\mathsf{T}_2 \cap \mathsf{T}_1$ are not countably generated over $\mathcal{O}^{(1)}$.

Proof. If C is a countable set, then $C \cup \mathcal{O}^{(1)}$ is Borel, so $\langle C \cup \mathcal{O}^{(1)} \rangle$ is analytic, hence not complete coanalytic. \square

The well-known analysis of coanalytic sets now gives the following:

4.6. *Theorem.* There is a sequence $(C_i : i \in \omega_1)$ of Borel clones such that:
 $i < j$ implies $C_i \subsetneq C_j$, $\bigcup_{i \in \omega_1} C_i = \mathsf{T}_2$, and:

For every analytic clone $C \subseteq \mathsf{T}_2$ there is $i < \omega_1$ such that $C \subseteq C_i$.

In other words: There is an increasing family of \aleph_1 many analytic clones below T_2 such that every analytic clone below T_2 is covered by a clone from the family.

A similar representation can be found for $\text{Pol}(\mathsf{T}_2)$, $\mathsf{T}_1 \cap \mathsf{T}_2$, etc.

Proof. By 2.3(1), we can find an increasing family of Borel sets $(B_i : i < \omega_1)$ such that $\mathsf{T}_2 = \bigcup_i B_i$. Clearly each clone $\langle B_i \rangle$ is analytic. By the boundedness theorem (2.3(2)) we know that for all i there is j with $\langle B_i \rangle \subseteq B_j$. Let $h : \omega_1 \rightarrow \omega_1$ be continuous and strictly increasing such that $\forall i : \langle B_i \rangle \subseteq B_{f(i)}$. Now the family $\{B_i : f(i) = i\}$ is as desired. \square

4.7. *Question.* Find a nice cofinal family in $\{\mathcal{C} : \mathcal{C} \subsetneq \mathsf{T}_2\}$. I.e., a nice family \mathcal{F} such that $\forall \mathcal{C} \subsetneq \mathsf{T}_2$ there is $\mathcal{C}' \in \mathcal{F}$ with $\mathcal{C} \subseteq \mathcal{C}'$.

Since we already have a family that covers all analytic clone, this question asks really: which nonanalytic clones are there below T_2 ?

4.8. *Question.* Can we get a family B_i as in the theorem where each B_i is generated by a single function?

4.9. *Question.* Analyze the interval $[\mathsf{T}_2, \text{Pol}(\mathsf{T}_2)]$.

REFERENCES

- [1] G.P. Gavrilov. Ueber funktionale Vollstaendigkeit in der abzählbar-wertigen Logik. *Probl. Kibernetiki*, 15:5–64, 1965.
- [2] Martin Goldstern and Saharon Shelah. Clones on regular cardinals. *Fundamenta Mathematicae*, 173:1–20, 2002. [arXiv:math.RA/0005273](#)
- [3] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.
- [4] Yiannis Nicholas Moschovakis. *Descriptive Set Theory*, volume 100 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Company, Amsterdam New York Oxford, 1980.
- [5] Michael Pinsker. Clones containing all almost unary functions. *Algebra Universalis*, to appear, 2004. [arXiv:math.RA/0401102](#)
- [6] R. Pöschel and L. A. Kalužnin. *Funktionen- und Relationenalgebren*, volume 15 of *Mathematische Monographien [Mathematical Monographs]*. VEB Deutscher Verlag der Wissenschaften, Berlin, 1979. Ein Kapitel der diskreten Mathematik. [A chapter in discrete mathematics].

DISCRETE MATHEMATICS AND GEOMETRY, ALGEBRA RESEARCH GROUP
 TU WIEN, WIEDNER HAUPTSTRASSE 8–10/104
 1040 WIEN, AUSTRIA (EUROPE)
E-mail address: Martin.Goldstern@tuwien.ac.at
URL: <http://www.tuwien.ac.at/goldstern/>